JOURNAL OF APPROXIMATION THEORY 78, 112–116 (1994)

Relations between Approximation Numbers and Entropy Numbers

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Communicated by Paul Butzer

Received May 8, 1992; accepted in revised form March 8, 1993

The paper deals with the interrelations of approximation numbers and entropy numbers of compact operators in quasi-Banach spaces.

1. INTRODUCTION AND RESULTS

Let A and B be two quasi-Banach spaces and let $T\colon A \to B$ be a compact mapping from A to B. Let U_A be the unit ball in A and let TU_A its image in B. Let $k \in \mathbb{N}$, then

(i) the kth entropy number e_k of T is the infimum of all $\varepsilon > 0$ such that there exists 2^{k-1} balls in B of radius ε which cover TU_A and

(ii) the kth approximation number a_k of T is the infimum of all numbers

$$\sup\{\|Tu - Lu \mid B\| : u \in U_A\}, \tag{1}$$

where L runs through the collection of all continuous linear mappings from A to B with rank $L < k$.

THEOREM. (i) *Let*

$$a_{2^{j-1}} \leqq c a_{2^j} \qquad \text{for some} \quad c > 0 \text{ and all } j \in \mathbb{N}. \tag{2}$$

Then it holds that

$$e_j \leqq C a_j \qquad \text{for some} \quad C > 0 \text{ and all } j \in \mathbb{N}. \tag{3}$$

(ii) *Let $f(j)$ be a positive increasing function on $\mathbb{N}$ with*

$$f(2^j) \leqq c f(2^{j-1}) \qquad \text{for some} \quad c > 0 \text{ and all } j \in \mathbb{N}. \tag{4}$$

0021-9045/94 $6.00

Then there exists a number $C > 0$ such that for all $n \in \mathbb{N}$ it holds that

$$\sup_{1 \leqq j \leqq n} f(j)\, e_j \leqq C \sup_{1 \leqq j \leqq n} f(j)\, a_j. \tag{5}$$

Remark 1. Recall that B is called a quasi-Banach space if it has all the properties of a Banach space with the possible exception of the triangle inequality which is replaced by its generalization

$$\|u + v \mid B\| \leqq c(\|u \mid B\| + \|v \mid B\|) \tag{6}$$

for some $c \geqq 1$ and all $u \in B$, $v \in B$ (of course, if $c = 1$ is admissible then B is a Banach space).

EXAMPLES. Of course, both $f(j) = j^{\varrho}$ and $f(j) = (\log(j+1))^{\varrho}$ for some $\varrho > 0$ and $j \in \mathbb{N}$ satisfy (4). Hence, we have

$$\sup_{1 \leqq j \leqq n} j^{\varrho} e_j \leqq C \sup_{1 \leqq j \leqq n} j^{\varrho} a_j, \qquad n \in \mathbb{N} \tag{7}$$

and

$$\sup_{1 \leqq j \leqq n} (\log(j+1))^{\varrho}\, e_j \leqq C \sup_{1 \leqq j \leqq n} (\log(j+1))^{\varrho}\, a_j, \qquad n \in \mathbb{N}. \tag{8}$$

Equation (7), restricted to Banach spaces, coincides with Carl's beautiful inequality; see [1, p. 294; 2, p. 96]. Our interest in the generalization (5) comes from "real life." In [3] we studied embeddings between function spaces of Besov–Hardy–Sobolev type in the so-called non-limiting case. The typical behaviour both of a_j and e_j is of power type, i.e., $a_j \sim j^{-\varrho}$ for some $\varrho > 0$. On the other hand, in the limiting case of compact embeddings of function spaces of Besov–Hardy–Sobolev spaces in Orlicz spaces of Trudinger type we have, for instance, $a_j \sim (\log(j+1))^{-\varrho}$ for some $\varrho > 0$; see [5] and the references given there. Then inequalities of type (8) are of interest. In other words, this piece of work is motivated by concrete examples characterized by $f(j) = j^{\varrho}$ or $f(j) = (\log(j+1))^{\varrho}$ and by the desire to shed more light on the related abstract background.

Remark 2. We collect some more or less known assertions connected with (3) and (5) and related counter-examples.

(i) When the Banach space B fails to have the approximation property then it may happen that a_j for $j \to \infty$ does not tend to zero; see [4, p. 32]. In that case (3) is trivial.

(ii) Let $T \neq 0$ and $e_j \leqq c 2^{-\varrho j}$ for some $c > 0$, $\varrho > 0$ and all $j \in \mathbb{N}$. Then T is of finite rank and, hence, $a_k = 0$ for large $k \in \mathbb{N}$; see, e.g., [2, p. 14]. In that case (3) is wrong.

(iii) Even more exotic cases violating (3) are known, see [2, p. 106], where one finds an example with $a_j \sim 2^{-j}$ and $e_j \sim 2^{-\sqrt{2j}}$.

2. Proof

Step 1. Iterative application of (6) yields

$$\left\| \sum_{j=1}^{k} y_j \mid B \right\| \leqq \sum_{j=1}^{k} 2^{\lambda(k+1-j)} \|y_j \mid B\| \tag{9}$$

for some $\lambda \geqq 0$. Let $L_j: A \to B$ such that

$$\operatorname{rank} L_j < 2^{j-1} \qquad \text{and} \qquad \|T - L_j\| \leqq 2a_{2^{j-1}}, \tag{10}$$

where $j \in \mathbb{N}$. By (9) and $L_1 = 0$ we have

$$\begin{aligned} \left\| Tx - \sum_{j=1}^{k-1} y_j \mid B \right\| &= \left\| (T - L_k)\,x + \sum_{j=1}^{k-1} [(L_{j+1} - L_j)\,x - y_j] \mid B \right\| \\ &\leqq 2^{\lambda} \|(T - L_k)\,x \mid B\| \\ &\quad + \sum_{j=1}^{k-1} 2^{\lambda(k+1-j)} \|(L_{j+1} - L_j)\,x - y_j \mid B\|, \end{aligned} \tag{11}$$

where $x \in U_A$ and $y_j \in B$. By (10) we have

$$\operatorname{rank}(L_{j+1} - L_j) < 2^{j+1} \qquad \text{and} \qquad \|L_{j+1} - L_j\| \leqq c a_{2^{j-1}} \tag{12}$$

for some $c > 0$. The first summand on the right-hand side in (11) can be estimated from above by $2^{\lambda+1} a_{2^{k-1}}$. As for the other summands in (11) we put $m_j = \mu 2^j (k-j)$ where $\mu > 0$ will be chosen later on. Let ε_{m_j} be the infimum of all $\varepsilon > 0$ such that there exists $2^{m_j - 1}$ balls in B of radius ε which cover the unit ball in the image of $L_{j+1} - L_j$. By (12) and elementary calculations these numbers can be estimated by

$$\varepsilon_{m_j} \leqq c_1 2^{-c_2 m_j / 2^j} = c_1 2^{-c_2 \mu (k-j)} \tag{13}$$

for some $c_1 > 0$ and $c_2 > 0$, see, e.g., [2, p. 21], when B is a real Banach space (there is no difficulty to extend the arguments given there to real or complex quasi-Banach spaces). By

$$\sum_{j=1}^{k-1} m_j = \mu 2^k \sum_{j=1}^{k-1} 2^{-(k-j)}(k-j) \leqq c\mu 2^k, \tag{14}$$

(11), the second part of (12), and (13) we arrive at

$$e_{c\mu 2^k} \leqq 2^{\lambda+1} a_{2^{k-1}} + c_1 \sum_{j=1}^{k} 2^{\lambda(k+1-j)} a_{2^{j-1}} 2^{-c_2\mu(k-j)}, \tag{15}$$

where $\mu > 0$ is at our disposal, c is the same as in (14), and c_1 and c_2 are appropriate positive numbers which are independent of μ.

Step 2. We prove part (i) of the Theorem. By (2) we have

$$a_{2^{j-1}} \leqq 2^{\kappa(k-j)} a_{2^{k-1}}, \qquad \text{where} \quad j = 1, ..., k \tag{16}$$

for some $\kappa > 0$. We insert (16) in (15) and choose μ sufficiently large. Then we get

$$e_{c 2^k} \leqq c' a_{2^{k-1}}, \qquad k \in \mathbb{N}, \tag{17}$$

where c comes from (14) and (15) and $c' > 0$ is an appropriate number. We may assume $c = 2^n$ for some $n \in \mathbb{N}$. By (2) and (17) we obtain

$$e_{2^{k+n}} \leqq c'' a_{2^{k+n}}, \qquad k \in \mathbb{N}. \tag{18}$$

Now (3) follows from (18), (2), and the monotonicity properties of a_j and e_j.

Step 3. We prove part (ii) of the Theorem. Let

$$a_j \leqq b f(j)^{-1} \qquad \text{for some} \quad b > 0 \text{ and } j = 1, ..., 2^{k-1}. \tag{19}$$

Then we can replace the a's in (15) by the right-hand side of (19). Now (4) is the direct counterpart of (2). Then by the respective counterparts of (16)–(18) we obtain

$$e_{2^{k+n}} \leqq c b f(2^{k+n})^{-1}, \qquad k \in \mathbb{N}, \tag{20}$$

where $c > 0$ is independent of b. We may choose

$$b = \sup_{j=1, ..., 2^{k+n}} f(j)\, a_j.$$

Again by (20) and the monotonicity properties of e_j, we arrive at (5).

Remark 3. See [2, pp. 96–100] for similar arguments connected with (7) in the case of Banach spaces.

Remark 4. We sketch a modified proof of (5). Extending the proof in [2, pp. 96–99] to quasi-Banach spaces we have (5) with $f(j) = j^{\alpha}$ where

$0<\alpha<\infty$. By (4) it follows that $2^{j\alpha}f(2^j)^{-1}$ is monotonically increasing for some $\alpha>0$. Hence by (5) with $f(j)=j^\alpha$ we have

$$\begin{aligned} 2^{k\alpha}e_k &\leqslant c \sup_{1\leqq j\leqq k} 2^{j\alpha}f(2^j)^{-1} \sup_{1\leqq j\leqq k} f(2^j)\, a_{2^j} \\ &\leqq c\,2^{k\alpha}f(2^k)^{-1} \sup_{1\leqq j\leqq k} f(2^j)\, a_{2^j}. \end{aligned}$$

But this coincides essentially with (20).

Acknowledgment

I wish to thank the referee. His or her comments improved the formulation of the theorem, and the modification of the proof suggested by him or her is briefly mentioned in Remark 4 at the end of the paper.

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